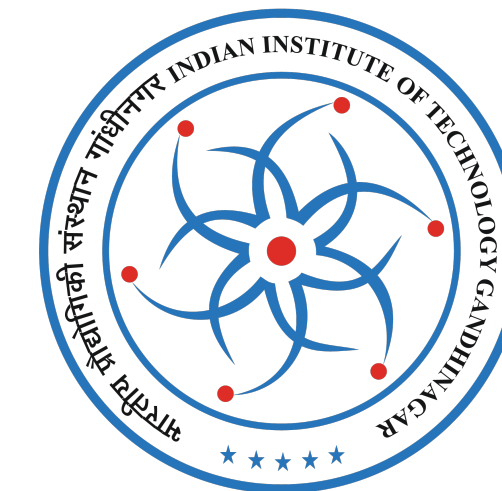


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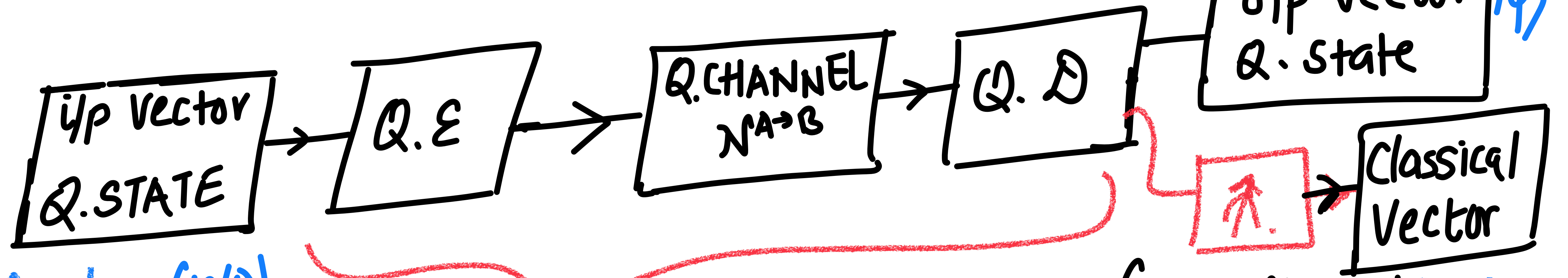
Decoupling theorem in quantum information theory: Application and Derandomization

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Analogous Quantum (Q) System



Vector-HILBERT SPACE

Vector $|\psi\rangle$
 $\in$
complex, norm.
complete vector
space (HILBERT SPACE)

Linear Transformation
(on the vector space of
LINEAR OPERATORS)

(Measurement) Random
Variable

Formally: Postulates from Quantum Mechanics.

I. Every quantum object (state of a particle) $\in$ HILBERT SPACE.

Quantum System (Postulates of QM):

1. Quantum states: Vectors/Matrices in Hilbert Space.

Typically: Quantum Analogue of classical pmf.

$\rho \geq 0$ (POS, Hermitian operator) s.t. $\text{Tr}(\rho) = 1$.

Quantum representation of a classical pmf:

$$(p_1 \dots p_{|x|})^T \xrightarrow{Q.} \text{diag}(p_1 \dots p_{|x|}).$$

— Pure state: $\text{rank}(\rho) = 1 \Rightarrow \rho = |x\rangle\langle x|$.

Mixed State: $\text{rank}(\rho) > 1$: DENSITY MATRIX.

Quantum System (Postulates of QM) :

Difference b/w classical and quantum states :

- Deterministic classical state: either 0 or 1 w.p. 1.

- Pure quantum state: superposition of 0 and 1:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \text{ s.t. } |\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C}.$$

$$|0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \quad |1\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}^T. \quad |\cdot\rangle \rightarrow \text{Dirac's ket notation for vectors.}$$

2 level quantum system $\rightarrow$ QUBIT system.

ENTANGLED STATE $\neq \sum_i |a_i\rangle^A \otimes |b_i\rangle^B$; e.g. $|\Phi\rangle^{\text{MES}} \triangleq \frac{|0\rangle^A |0\rangle^B + |1\rangle^A |1\rangle^B}{\sqrt{2}}$

CORRELATED STATE: $\frac{|0\rangle^A |0\rangle^B \otimes |0\rangle^A |0\rangle^B + |1\rangle^A |1\rangle^B \otimes |1\rangle^A |1\rangle^B}{2}$

Quantum System (Postulate of QM) :

Composition of Quantum systems $\equiv$ TENSOR PRODUCT.

Sys 1 : $\mathcal{H}_{A_1}$, Sys 2 : $\mathcal{H}_{A_2}$..., Sys k : $\mathcal{H}_{A_k}$

Composite sys $\triangleq \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \dots \otimes \mathcal{H}_{A_k}$.

— Fixing a basis $\equiv$ Tensor Product: Kronecker Product.

$$\text{e.g. } \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \triangleq \begin{bmatrix} a_1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ a_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{bmatrix}; \quad M \otimes N \triangleq \begin{bmatrix} m_{11}[N], \dots, m_{1d}[N] \\ \vdots \\ m_{kd}[N] \end{bmatrix}$$

Quantum System (Postulates of QM)

③ Every CLOSED quantum system evolves according to time dependent Schrödinger's Equation.

⇒ "Noiseless" channel: (UNITARY OPERATOR, $U: U^\dagger U = I$)
($\rho \xrightarrow{U} U\rho U^\dagger$)

⇒ "Noisy" Channel: LINEAR, COMPLETELY POSITIVE
TRACE PRESERVING (CPTP) MAP.

Classical Analogy: i/p pmf $\xrightarrow{\mathcal{N}}$ o/p PMF $\equiv \mathcal{N}$: col. stochastic

Quantum: i/p ρ ($\rho \geq 0, \text{Tr}(\rho) = 1$) $\xrightarrow{\mathcal{N}}$ o/p $\sigma = \mathcal{N}(\rho) \geq 0, \text{Tr}(\mathcal{N}(\rho)) = \text{Tr}(\rho) = 1$

Quantum System (Postulates of QM) :

③. Quantum Channel: TP (Trace Preserving) :

$$\text{Tr}[\mathcal{N}(\rho)] = \text{Tr}(\rho)$$

CP (Complete Positive) $\triangleq \left(\mathbb{1}_{n \times n} \otimes \mathcal{N} \right) \left(\tau^{A'A} \right) \geq 0 \quad \forall \tau \geq 0, \quad n \geq 1.$

④. Measurement Postulate: MEASUREMENT $\Rightarrow$ o/p as R.V.
 $\equiv$ Operators $\{\lambda_m\}_{m=1}^d$ s.t. $\boxed{\text{o/p } M} \in \{1, \dots, d\}$ and
 $\boxed{P(M=m) = \text{Tr}(\lambda_m \rho)}$; $\sum_m \lambda_m = \mathbb{1} \Rightarrow \sum_m P(M=m) = \text{Tr}(\rho) = 1.$

Last review point: Quantum Channels:

Any Quantum Channel ($\equiv$ CPTP map) can equivalently be represented as:

①. Choi Matrix: $J_N \stackrel{AB}{=} \left(\mathbb{1}_A \otimes N_{A \rightarrow B} \right) \Phi^{MES} \geq 0$

$\equiv$ Classical Transition matrix

$$N(e) = \text{Tr} [J_N (p_A^T \otimes \mathbb{1}_B)].$$

②. Stinespring: $N \equiv U_N^{AE}$: $N(e) = \text{Tr}_E [U_N (p_A \otimes \delta_E) U_N^\dagger]$.

Quantum Entropies:

1. $H(A)_{\rho_A} \triangleq -\text{Tr}_A [\rho_A \log \rho_A]; \rho = \sum_i \lambda_i |i\rangle\langle i|, \lambda_i \geq 0, \sum_i \lambda_i = 1$

[For any $M \geq 0, M = \sum_i m_i p_i \Rightarrow \log M \triangleq \sum_i \log m_i p_i$]

$\rho = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{|A|}); H(\rho) = H(\vec{\lambda}) \leftarrow \text{Shannon's.}$

2. $H(A|B)_{\rho_{AB}} \triangleq H(AB)_{\rho} - H(B)_{\rho}; \rho_B \triangleq \text{Tr}_A(\rho_{AB}).$

$$\rho_B \triangleq \text{Tr}_A(\rho_{AB}) \triangleq \sum_{i=1}^{|A|} (\langle a_i|^A \otimes \mathbb{1}^B) \rho_{AB} (\lvert a_i \rangle^A \otimes \mathbb{1}^B),$$

for any o.n.b. $\{\lvert a_i \rangle\}_{i=1}^{|A|}$.

ex: $\text{Tr}_A(\sigma_A \otimes \sigma_B) = \text{Tr}(\sigma_A) \cdot \sigma_B \leftarrow \text{"Marginal"}$.

Abstractly: $\boxed{\langle \text{Tr}_A(M_{AB}), N^B \rangle \triangleq \langle M_{AB}, 1_A \otimes N_B \rangle}$

We'll be using ONE-SHOT CONDITIONAL ENTROPIES:
(Derived from Rényi & Divergences).

$$\textcircled{1}. H_2(A|B)_f \triangleq -\log \left[\inf_{\substack{\sigma_B \geq 0: \\ \text{Tr}(\sigma) = 1}} \text{Tr} \left[(1_A \otimes \sigma_B)^{-1/2} \rho_{AB} \right]^2 \right]$$

[If B is trivial, i.e., $|B|=1$, $\rho_B = 1 = H_2(A)$: Collision entropy]

$$\textcircled{2}. H_{\min}(A|B)_f \triangleq -\log \inf \{ \text{Tr}(\sigma^B) : \sigma^B \geq 0, \rho_{AB} \leq 1^A \otimes \sigma_B \}$$

$$[B: \text{trivial} \Rightarrow H_{\min}(A)_f = -\log \lambda_{\max}(\rho)]$$

$$\rightarrow H_{\min}(A|B) \leq H_2(A|B)$$

Important Technical Observation:

→ No AEP for $H_2(A|B)_p$ and $H_{\min}(A|B)_p$, i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_{2/\min}(A^n|B^n)_{p^{\otimes n}} \neq H(A|B)_p.$$

→ Reason: Slight perturbation in p can make $H_{2/\min}$ diverge.

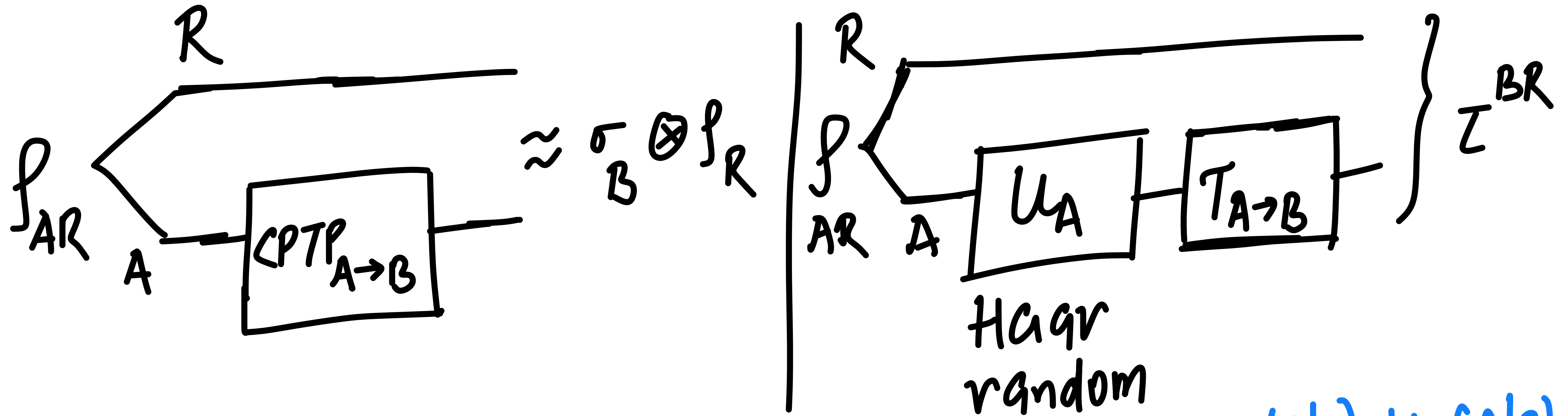
Fix: Smoothing of p : SHAVING off small e.values of p :

$$H_2^\varepsilon(A|B)_p \triangleq \inf_{p' \in \mathcal{B}^\varepsilon(p)} H_2(A|B)_{p'} : \mathcal{B}^\varepsilon(p) \triangleq \{p' : \|p' - p\|_1 \leq \varepsilon\}.$$

Leads to AEP: $\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} H_2^\varepsilon(A^n|B^n)_{p^{\otimes n}} = H(A|B)_p$

[Tomamichela thesis, 2012]

DECOUPLING THEOREM (Dupuis Thesis, 2010)



Thm: $\mathbb{E}_{U_A \sim \text{Haar}} \left\| \left(\tau^{A \rightarrow B} \otimes \mathbb{1}^R \right) \rho_{AR} - \omega^B \otimes \rho^R \right\|_1 \leq 2^{-\frac{1}{2} H_2(A|B) - \frac{1}{2} H_2(A|R)}$

where $\tau^{A \rightarrow B} \otimes \mathbb{1}^R = \omega^{BR}$

Trick: $\text{Tr}_{A'} \left[\left(\tau^{A \rightarrow B} \otimes \mathbb{1}^{A'} \right) \Phi^{AA'} \right]$

Decoupling Theorem : Smoothed Version :

$$\mathbb{E}_{U \sim \text{Haar}} \left[\|T(U \cdot f) - \omega^B \otimes f^R\|_1 \right] \leq 2^{-\frac{1}{2} [H_2^\varepsilon(A|B)_\omega + H_2^\varepsilon(A|R)_f]} + 8\varepsilon$$

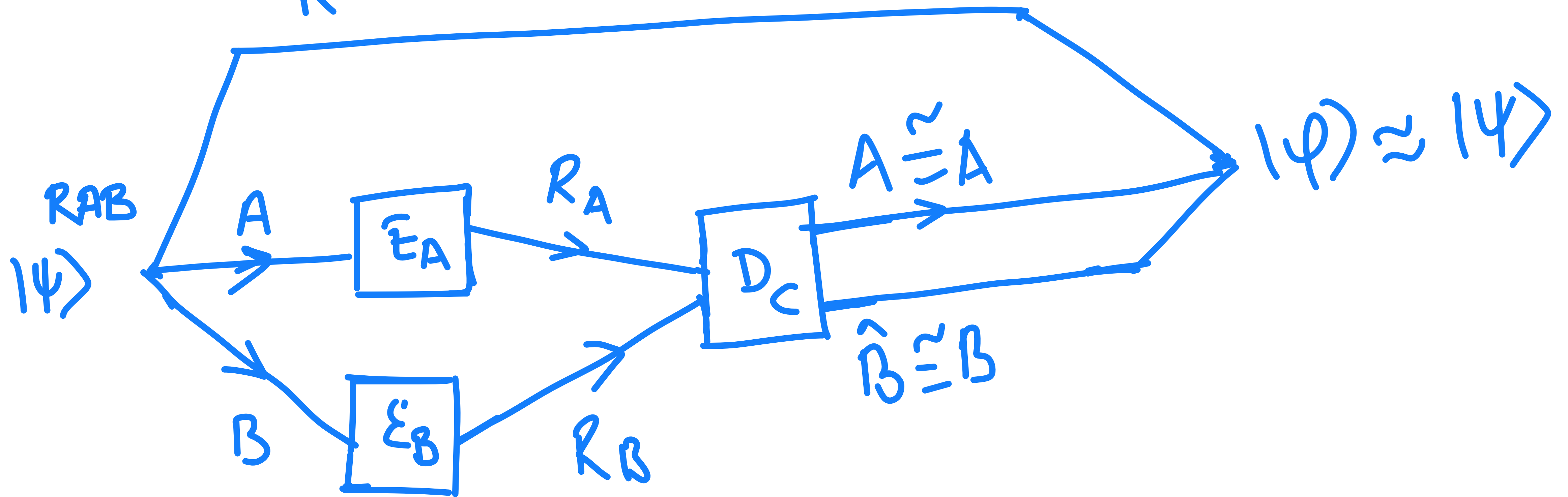
$$\rightarrow \underline{\text{AEP}} : \mathbb{E} \left\| T^{\otimes n} (U \cdot f^{\otimes n}) - \omega_{B^n} \otimes f_R^{\otimes n} \right\|_1$$

$$\leq 2^{-\frac{n}{2} [H(A|B)_\omega + H(A|R)_f + O(\varepsilon)]} + 8\varepsilon$$

for n large enough.

Applications: Fully Quantum Slepian Wolf (FQSW);
 Achievability for pt.-pt. quantum channel capacity.

I. FQSW: $A = A_1 \otimes A_2$, $B = A_2$, $T^{A \rightarrow A_2} \triangleq \text{Tr}_{A_1}$.

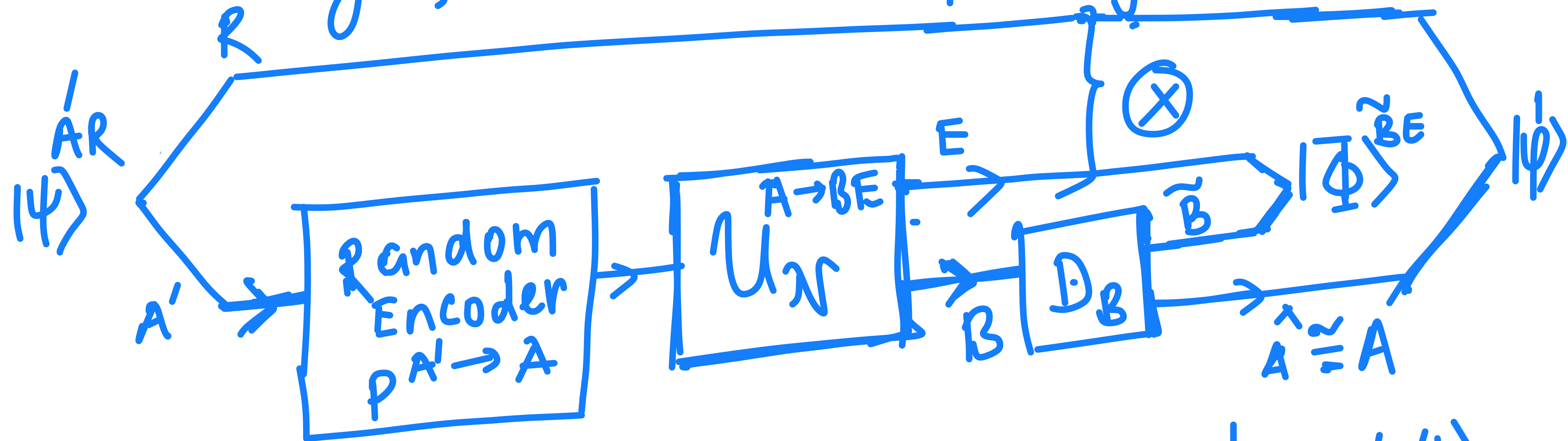


FQSW: (IID version):

- Alice performs Schumacher comp. on A & tx at rate $R_A \geq H(A)$ qubits.
- Bob performs decoupling with $T = \text{Tr}_B$ & tx at rate $R_B \geq -H(B|R)$
- Sum rate: $R_A + R_B \geq H(A) - H(BR) + H(R) = H(AB)$.

This is same as classical SW.

II. Achievability for channel capacity:



Task: $\max_{P, D_B} \log |A| \triangleq Q$ s.t. $|\psi'\rangle \approx |\psi\rangle$.

$$\text{FREE} \leftarrow V_{B \rightarrow \hat{A} \tilde{B}} \varphi_{REB} \approx \varphi'_{RA} \otimes \varphi_{\tilde{B} \tilde{B}} \leftarrow \text{Uhlmann's theorem.} \\ \approx |\psi\rangle_{RA'}$$

Capacity : Given by decoupling :

Take $P_{AR} = \mathbb{E}_{MES} \xrightarrow{\text{Proj. on Haar random } A} H(A|R) = +\log |A|$

$$\Rightarrow \sqrt{|A|} 2^{-H(A|B)} \omega \leq \delta \Rightarrow \log |A| \leq -H(A|B) + \delta$$

$$\Rightarrow Q = \max \log |A| \approx -H(A|B) + \delta; I_c(A>B) \stackrel{\Delta}{=} -H(A|B)$$

How sharp is decoupling?

Ans. Find $P_{U \sim \text{Haar}} (|f(u) - \mathbb{E} f(u)| > \delta) \leq 2e^{-|A|\delta^2/4L^2}$;
[$f = L$ Lipschitz] [Dudnis theo]

Issues: Haar random U is exponentially expensive
– $f(u)$ is not a poly. in entries on U .

Fix: Replace Haar random U via $U \sim \text{approximate}$
unitary t design, efficient implementation
for $t = \text{poly}(n)$.

– $\|f(u)\|_1 \leq \|f(u)\|_2^2$.

↳ WEIGHTED CAUCHY-SCHWARZ.

Rest of the talk: Proof of Decoupling (in $\mathbb{E}$) + Derandomization.

Key Elements in Proof (follow via linear algebraic analysis):

$$\textcircled{1}. \forall M: M^T = M: \|M\|_1 \leq \sqrt{\text{Tr}(\sigma) \text{Tr}(\sigma^{-1/4} M \sigma^{-1/4})^2} \\ = \|\sigma^{-1/4} M \sigma^{-1/4}\|_2. \\ \text{with } \sigma > 0.$$

$$\textcircled{2}. \mathbb{E}_{U \sim \text{Haar}} \left(U^{\otimes 2} M_{AA'} (U^T)^{\otimes 2} \right) = \alpha \mathbb{1}^{AA'} + \beta \widetilde{F}^{AA'} \\ \text{[} A' \cong A \text{]}; F = |i\rangle_A |j\rangle_{A'} \mapsto |j\rangle_A |i\rangle_{A'} \quad (\text{SWAP})$$

Non-trivial part : $\int_{\text{Haar}} U^{\otimes 2} M(U) U^{\otimes 2} dU$: SCHUR-WEYL DUALITY.

Towards a "friendly" version of Schur-Weyl Duality:
- Want to find $M_{\mu_H}^{(k)}(O) \triangleq \mathbb{E}_{U \sim \mu_H} [U^{\otimes k} O(U) U^{\otimes k}]$
+ O. (For us $k=2$).

Def: k^{th} order commutant of a set of matrices
 $S \equiv \text{Comm}(S, k) \triangleq \{ A : A B^{\otimes k} = B^{\otimes k} A, \forall B \in S \}$
Note: $\text{Comm}(S, k) \leftarrow$ vector subspace.

Fact 1: $M_{\mu_H}^{(k)}(\theta) \in \text{Comm}(U_{d,k})$; if $A \in \text{Comm}(U_{d,k})$
 then, $M_{\mu_H}^{(k)}(A) = A$.

Pf: $V^{\otimes k} M_{\mu_H}^{(k)}(\theta) = \mathbb{E} \left[(VU)^{\otimes k} \sigma U^{\dagger \otimes k} \right]$
 $= \mathbb{E} \left[U^{\otimes k} \sigma (V^\dagger U)^{\dagger \otimes k} \right] \leftarrow \text{Rotation Inv. of } \mu_H$
 $= M_{\mu_H}^{(k)}(\theta) V^{\otimes k} \quad \square$

If $A \in \text{Comm}(U_{d,k}) \Rightarrow M_{\mu_H}^{(k)}(A) = \mathbb{E} \left(\overset{\otimes k}{U} A \overset{\otimes k}{U}^{\dagger} \right) = A$

Fact 1 $\Rightarrow M_{\mu_H}^{(k)}(0)$ is orthogonal projection
 onto $\text{Comm}(U_d, k)$ w.r.t. HS inner Product.

$$\equiv M_{\mu_H}^{(k)}(0) = \sum_{i=1}^{|\text{Comm}|} \langle P_i, 0 \rangle P_i,$$

$\{P_i\}_{i=1}^{|\text{Comm}|}$ are o.n.b. of $\text{Comm}(U_d, k)$.

Schur-Weyl Duality (Formally):

$$\text{Comm}(U_d, k) = \text{span} \{ \text{Permutation matrices on } \text{Sym. gp} : V_d(\pi_i) \}.$$

$$V_d(\pi): V_d(\pi) |\psi_1\rangle \otimes \dots \otimes |\psi_k\rangle = |\psi_{\pi^{-1}(1)}\rangle \otimes \dots \otimes |\psi_{\pi^{-1}(k)}\rangle$$

$\forall \pi \in S_k$ (Symm gp on k elements)

Thus, Schur-Weyl Duality $\Rightarrow \mathcal{M}_{\mu_H}^{(2)}(0) = \alpha \underline{1} + \beta F$.

α, β are calculated by solving:

$$\text{Tr}(0) = \alpha d^2 + \beta d \quad \text{and} \quad \text{Tr}(F \mathcal{M}_{\mu}^{(2)}(0)) = \text{Tr}(F0) = \alpha d + \beta d^2$$

Putting it all together to finish the proof:

$$\textcircled{1}. \|\tau(U_A \cdot f_{AR}) - \omega_B \otimes f_R\|_1^2 \leq \|\sigma_B^{-1/4} \cdot \tau(U_A (\Sigma_R^{-1/4} f_{AR})) - \sigma_B^{-1/4} \cdot \omega_B \otimes \Sigma_R^{-1/4} f_R\|_2^2$$

for $\sigma_B > 0, \Sigma_R > 0$; $\tilde{\tau}(\cdot) \triangleq \sigma_B^{-1/4} \tau(\cdot) \sigma_B^{-1/4}$,
 $\tilde{\omega}_B \triangleq \sigma_B^{-1/4} \omega_B \sigma_B^{-1/4}$; $\tilde{f}_R \triangleq \Sigma_R^{-1/4} f_R \Sigma_R^{-1/4}$.

Now to compute: $\mathbb{E} \|\tilde{\tau}(U_A \tilde{f}_{AR}) - \tilde{\omega}_B \otimes \tilde{f}_R\|_2^2$.

$$\rightarrow \mathbb{E} \|\tilde{\tau}(U_A \cdot \tilde{p}_{AR}) - \tilde{\omega}_B \otimes \tilde{p}_R\|_2^2 = \mathbb{E} \|\tilde{\tau}(U_A \cdot \tilde{p}_{AR})\|_2^2 - \|\tilde{\omega}_B\|_2^2 \|\tilde{p}_R\|_2^2;$$

$$\mathbb{E} \|\tilde{\tau}(U_A \cdot \tilde{p}_{AR})\|_2^2 = \text{Tr} \left[\tilde{p}_{AR}^{\otimes 2} \left(\mathbb{E} \{ U_A^{\otimes 2} \tilde{\tau}^{\otimes 2}(F_B) \} \otimes F_R \right) \right]$$

$$(\text{Using: } \text{Tr}[(M \otimes N)F] = \text{Tr}[MN],$$

$$\langle \tau(p), \sigma \rangle = \langle p, \tau^+(\sigma) \rangle)$$

$$\text{Schur-Weyl Duality} \Rightarrow \mathbb{E} \left[U^{\otimes 2} \tilde{f}^{\otimes 2}(F_B) \right] = \alpha \mathbb{1} + \beta F^{AA'}$$

$$\text{and } \alpha |A|^2 + \beta |A| = |A|^2 \|\tilde{\omega}^B\|_2^2; \alpha |A| + \beta |A|^2 = |A|^2 \|\tilde{\omega}^{AB}\|_2^2$$

$$\left(\text{Finally using: } \frac{1}{|A|} \leq \frac{\|\tilde{\omega}^{AB}\|_2^2}{\|\tilde{\omega}^B\|_2^2} \leq |A| \text{ (for any } \tilde{\omega}^{AB} \geq 0) \right)$$

$$\begin{aligned} \text{We get: } \mathbb{E} \|\tilde{T}(U_A \tilde{f}_{AR})\|_2^2 &= \text{Tr} \left[\tilde{f}_{AR}^{\otimes 2} (\alpha \mathbb{1} + \beta F) \otimes F \right] \\ &= \alpha \|\tilde{f}_R\|_2^2 + \beta \|\tilde{f}_{AR}\|_2^2 \end{aligned}$$

Substing α, β and collecting all the terms:

$$\mathbb{E} \| \tau(u_A \cdot \rho_R) - \omega_B \otimes \rho_R \|,$$

$$\leq \frac{-\frac{1}{2} H_2(A|B) \omega^{-\frac{1}{2}} H_2(A|R) \rho}{2}$$

□

Towards smoothing:

Perturb τ to $\hat{\tau}$ s.t. $\max_{\psi_{AA'}} \| (\tau \otimes \mathcal{I} - \hat{\tau} \otimes \mathcal{I}) \psi \|_1 \leq \varepsilon.$

Using this with $f'_{AR} \in \mathcal{B}^\varepsilon(p_{AR})$, $w'_{AB} \in \mathcal{B}^\varepsilon(w_{AB})$
 that achieves equality in $H_2^\varepsilon(A|B)$, $H_2^\varepsilon(A|R)$,
non-trivially one can show:

$$\begin{aligned} & \mathbb{E} \| \tau(u \cdot f_{AR}) - w_B \otimes p_R \|_1 \\ & \leq 2^{-\frac{1}{2} H_2^\varepsilon(A|B)} 2^{-\frac{1}{2} H_2^\varepsilon(A|R)} + 8\varepsilon. \end{aligned}$$

(Dupuis
et al.,
2014).

Measure Concentration / Tail Probability:

Notation: $f(u) \triangleq \| \tau(u; p_{AR}) - \omega^B \otimes p^R \|$,

$$\mu \triangleq 2^{-\frac{1}{2} H_2(A|B)} \omega^{-\frac{1}{2} H_2(A|R)} p$$

→ Naively: Markov's $\Rightarrow \mathbb{P}(f(u) > s) \leq \mu/s$

But: this doesn't say how sharp f is conc.
around μ & above tail doesn't
necessarily decay fast!

→ Need finer analysis for $\mathbb{P}(|f(u) - \mu| > s)$.

→ Use Levy's Lemma for Lipschitz cont. function.

Challenge: Find a tight Lipschitz const. for f .

$$(L) : \frac{|f(u) - f(v)|}{\|u - v\|_2} \leq L_f$$

Turns out (after some work) : $L_f \leq 2\sqrt{\|g\|_\infty} \leq \sqrt{2} \frac{\epsilon}{H_{\min}(A)^{1/4}}$

Levy's Lemma: $P(|f(u) - \mu| > \delta) \leq e^{-|\mathcal{A}|\delta^2 / 4L_f^2}$

$\Rightarrow \boxed{P(f(u) \geq \mu + \delta) \leq e^{-c|\mathcal{A}|\delta^2 / 2^{H_{\min}(A)} \epsilon}} \leftarrow \text{(Duguis Thesis)}$

DERANDOMIZATION :

Prove a high probability decoupling /
Find tail probability of decoupling function

when $U_A \sim \text{Haar} \rightarrow U_A \sim \text{appx. unitary } t\text{-design.}$

Essentially: Replacing IID RVs by t -wise II and still
prove similar tail bounds & expectation.

δ -Appx. Unitary t -designs: Ensemble of finite # unitaries with some measure (gen. uniform) s.t. for any poly. $p(U)$ in the entries of U of deg. at most ' t ' satisfy:

$$\left| \mathbb{E}_{U \sim \text{unif design}} p(U) - \mathbb{E}_{U \sim \text{Haar}} p(U) \right| \leq \frac{\delta}{d^t}.$$

(for any given $\delta > 0$). Ex: $t=2,3 \leftarrow$ Clifford gt.

→ For $t = \text{poly}(\log d)$: these designs have efficient implementations (Brandao et al., 2010, Sen, 2018)

Crucial: To replace Haar rand. U via $U \sim \text{design}$

$p(U) \leftarrow \text{poly. in } U$.

$\|M\|_1 = \text{Tr} \sqrt{MM^T} \leftarrow \text{Not a poly.}$

So we use our earlier trick of wt. Cauchy-Schwarz

$$\|f(U)\|_1 \leq \|\tilde{f}(U)\|_2^2 \left[\|M\|_2^2 = \text{Tr}[MM^T] \right]$$

Our Results:

$$\textcircled{1}. \mathbb{E}_{U \sim \mu_{\text{design}}} f(U) \leq 2^{-\frac{1}{2} H_2^\varepsilon(A|R)} 2^{-\frac{1}{2} H_2^{\varepsilon, \delta}(A|B)_\omega} \triangleq \mu_{\text{design}}$$

(i). To obtain the $\mathbb{E}(\cdot)$ in terms of smooth entropies (to apply AEP in asymptotic settings) we come with a new variant of $H_2^\varepsilon(\cdot|\cdot)$, where ε is to smooth the states and δ is to smooth τ so that the weights in Wt. Cauchy-Schwarz can be controlled.

②. Concentration / Tail Probability for Decoupling with design :

$$\mathbb{P}(f(u) > \mu_{\text{design}} + 14\sqrt{\varepsilon} + 2\eta) \leq 7 2^{-a\eta^2};$$

$u \sim \text{design}$

where: $a \triangleq |A| 2^{-(1+\delta)} H_{\max}^{\varepsilon}(B)_{\omega} + H_2^{\varepsilon}(A|R) \leftarrow \text{Lip. const.}$

(of an intermediate $f^{\eta} g$, obtained by perturbing τ and considering the smoothing).

Design parameter $t = 8 a \eta^2$. $[f \xrightarrow{\text{C.S.}} \tilde{f} \xrightarrow[\tau]{\text{Pert.}} g]$

3. We also prove AEP for $\|H\|_2^{\varepsilon, \delta}(A|B)_\omega$, $\|H\|_{\max}^\varepsilon(B)_\omega$ i.e.

$$\lim_{\varepsilon, \delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \|H\|_2^{\varepsilon, \delta}(A^n|B^n)_{\omega^{\otimes n}} = H(A|B)_\omega \text{ and}$$

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \|H\|_{\max}^\varepsilon(B^n)_{\omega_B^{\otimes n}} = H(B)_\omega.$$

$$H_{\max}^\varepsilon(B)_\omega \triangleq 2 \log \min_{\sigma \geq 0, \sigma \in \mathcal{B}^\varepsilon(\rho)} \text{Tr} \sqrt{\rho}; \quad \|H\|_{\max}^\varepsilon(B)_\omega \triangleq \log \|\tilde{\omega}_\varepsilon\|_\infty$$

$\omega_\varepsilon \geq 0 \leftarrow$ zeroing out smallest e-values of ω that sum to $\leq \varepsilon$.

& $\omega_{\varepsilon, \delta}^B > 0$ than $\frac{1}{2} \frac{1}{(1+\delta) \|H\|_{\max}^\varepsilon(B)_\omega} \|\omega\|_1 - \omega$ that are smaller.

With $\omega_{\epsilon, \delta}^B$ so defined:

$$\|H\|_2^{\epsilon, \delta}(A|B)_\omega \triangleq -\log \min \|(\omega_{\epsilon, \delta}^B)^{-1/4} \cdot \eta^{AB}\|_2^2, \text{ where}$$

min. is over all PDS η^{AB} s.t. $\|\eta^{AB} - \omega^{AB}\|_1 \leq \epsilon$

and $\forall |v\rangle^{AB} \in \text{supp}\{\eta\}$ satisfy

$$\|\{\mathbb{1}^A \otimes \Pi_{\text{supp}(\omega_{\epsilon, \delta}^B)}\}|v\rangle\|_2^2 \geq 1 - \epsilon.$$

Applications: FQSW with design in Physics:

- ①. Efficient relative thermalization; ②. Hayden-Preskill toy model of Black hole as information mirror.

CONCLUSION :

①. Decoupling - widely applicable

Following works :

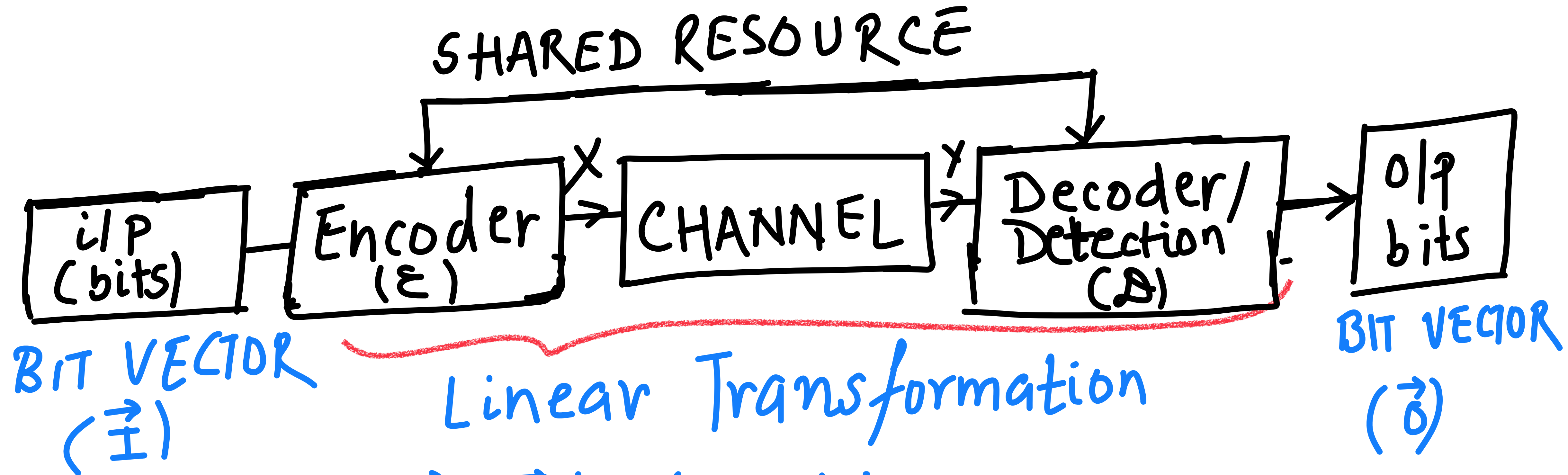
- Privacy amplification & decoupling without smoothing.
(Dupuis, 2021).
- A new multi-sender decoupling thm. for and simultaneous decoding for Quantum MAC
(Chakraborty, Nema & Sen, 2021).
- Decoupling by local random unitaries without simultaneous smoothing & app. to multi-user settings. (Colomer & Winter, 2024).

Thank You & Questions...

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CLASSICAL COMMUNICATION SYSTEM:



GOAL: $\|\vec{I} - \vec{O}\|$ is minimum.

CHANNEL: Linear Map, a transition matrix.

Noisy Channel: $\{p_{y|x}\}_{y|x \in \mathcal{X}}$: Column stochastic matrix.